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Gravitational Field of a Spin-Polarized Cylinder in the Einstein-Cartan Theory of Gravitation

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The gravitational field produced by an infinite static cylinder with a net spin polarization along its symmetry axis is considered, in the context of the Einstein-Cartan theory of gravitation. An exact solution is obtained, which shows that the cylinder's spin angular momentum gives rise to magneticlike components in the gravitational field outside.

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It is well known that, in the context of general relativity (GR), the orbital angular momentum of a rotating mass distribution gives rise to off-diagonal terms in the metric which are responsible for the magneticlike effects coming under the heading of the "dragging of inertial frames."[1] Among the first solutions of Einstein's field equations exhibiting such effects, the Lense-Thirring[2] metric gives an approximate representation of the gravitational field of a rigidly rotating sphere, while van Stockum's[3] exact solution describes the field of a rigidly rotating infinite cylinder of dust.

The question can then be naturally raised whether a gravitating source with a net intrinsic or spin angular momentum can produce similar effects. Any piece of matter with the spin of its particles aligned along a particular direction would be an example of such a source.

In this note I present the answer to the above question in terms of an exactly soluble model of a cylindrically symmetric distribution of polarized matter, constructed and interpreted on the basis of a theory of gravitation which has attracted a lot of interest recently, namely, the so-called Einstein-Cartan(-Sciama-Kibble) (EC) theory.[4] The most attractive feature of this theory consists of providing a geometric framework for the description of intrinsic or spin angular momentum without violating the spirit and the predictions of GR. (The reference just cited contains an extensive discussion of this point.)

The additional geometric feature of the space-time manifold which distinguishes EC theory from GR is the contorsion or defect tensor field $K^{\alpha}{}_{\beta\gamma}$. The latter is determined by the distribution of spin density, described by the tensor field $S^{\alpha}{}_{\beta\gamma}$, via the algebraic relation

$$T^{\alpha}{}_{\beta\gamma}+2\delta^{\alpha}{}_{[\beta}T_{\gamma]}=\kappa S^{\alpha}{}_{\beta\gamma}, \tag{1}$$

where $T^{\alpha}{}_{\beta\gamma}=2K^{\alpha}{}_{[\beta\gamma]}$ is the torsion tensor, $T_{\alpha}=T^{\beta}{}_{\alpha\beta}$, $\kappa=8\pi G/C^2$ is the GR coupling constant, but units will be chosen such that $c=1=\kappa$, and square brackets indicate antisymmetric combination of indices. Thus, torsion and contorsion are present only in regions of nonvanishing spin density. As in GR, the magnitude of a vector does not change upon parallel transport in EC space time. Consequently, the linear connection $\Gamma^{\alpha}{}_{\beta\gamma}$ of the manifold is determined by the metric and contorsion fields only, i.e., $\Gamma^{\alpha}{}_{\beta\gamma}=\gamma^{\alpha}{}_{\beta\gamma}-K^{\alpha}{}_{\beta\gamma}$, where $\gamma^{\alpha}{}_{\beta\gamma}$ is the usual Levi-Civita part constructed from $g_{\alpha\beta}$ and its derivatives.[5] All other geometric quantities are constructed as in GR, while the field equations read

$$R_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}R=\kappa t_{\alpha\beta} \tag{2}$$

reflecting the affinity of EC theory to GR. Equation (2) also reflects the difference between the two theories in that the canonical stress-energy tensor $t_{\alpha\beta}$ is in general asymmetric. The antisymmetric part of Eq. (2), however, is nothing but a different expression of the equation for conservation of angular momentum,

$$(\nabla_\gamma + T_\gamma) S^\gamma{}_{\alpha\beta} = 2t_{[\beta\alpha]}. \tag{3}$$

It can, thus, be subtracted from Eq. (2), leaving the symmetric part

$$R_{(\alpha\beta)} - \tfrac{1}{2} g_{\alpha\beta} R = \kappa t_{(\alpha\beta)}, \tag{4}$$

as the field equations proper.

I now consider a spin-polarized medium occupying the region $0 \le r \le R$ of a coordinate system in which the metric reads

$$ds^2 = -e^{2\nu}dt^2 + r^2 e^{-2\nu} d\varphi^2 + e^{2(\mu-\nu)}(dr^2 + dz^2), \tag{5}$$

where μ, ν are functions only of r, and (r, φ) represent polar coordinates. I assume that the particles of this medium describe the lines of constant (r, φ, z) and have their spins oriented in the direction of the positive z axis. I further assume that their spin density can be described classically, i.e., $S^\alpha{}_{\beta\gamma} = u^\alpha S_{\beta\gamma}$, with $S_{\alpha\beta}u^\beta = 0$, u^α being the particles' four-velocity. In the orthonormal frame $(e_0, e_1, e_2, e_3) = (e^{-\nu}\partial_t, e^{\nu-\mu}\partial_r, r^{-1}e^{\nu}\partial_\varphi, e^{\nu-\mu}\partial_z)$, these assumptions can be expressed as $u^\alpha = \delta_0{}^\alpha$, $S_{\alpha\beta} = 4S(r)\delta_{[\alpha}{}^1\delta_{\beta]}{}^2$. Equation (1), then, shows that $T^0{}_{12} = -T^0{}_{21} = 2S$ are the only nonvanishing components of the torsion tensor, while $T_\alpha = 0$.

The assumptions employed so far render the structure of the space-time region under consideration cylindrically symmetric and static. This is reflected in the fact that the tensor fields which determine the space-time structure, namely $g_{\alpha\beta}$, u^α, and $S_{\alpha\beta}$, are independent of φ, z, and t, while the velocity of the particles has vanishing projection in the surfaces of constant t. Technically speaking, the Lie derivative of the fundamental tensor fields $g_{\alpha\beta}$ and $K^\alpha{}_{\beta\gamma}$ along the vectors ∂_φ, ∂_z, and ∂_t vanishes, the latter being parallel to the timelike vector field u^α and hypersurface orthogonal.

Two further assumptions were made in order to obtain the solution to be presented shortly (with details to be published elsewhere). It was firstly assumed that the spin density is conserved, i.e., that $\dot{S}_{\alpha\beta} \equiv u^\gamma \nabla_\gamma S_{\alpha\beta} = 0$. It then follows from Eq. (3) that $t_{\alpha\beta} = t_{(\alpha\beta)}$, while the field equations (4), when written out explicitly, demand that $t_{\alpha\beta} = \mathrm{diag}(m, p_1, p_2, p_3)$. This form of $t_{\alpha\beta}$ permits our calling m the rest energy density and p_1, p_2, and p_3 the principal components of stress in the direction of r, φ, and z, respectively. Secondly, it was assumed that $p_1 = p_2$, since the cylindrical symmetry of the problem demands that the medium should be isotropic in the directions transverse to the z axis.

The last assumption, however, can be justified on more physical grounds, based on recent results of Ray and Smalley.[6] These authors have used a Lagrangian variational principle to obtain a stress-energy tensor for a perfect fluid in EC space-time, which takes into account the contribution of spin to the specific internal energy, ϵ, of the fluid. In our notation, the Ray-Smalley stress-energy tensor reads

$$t_{(\alpha\beta)} = \rho(1 + \epsilon + p/\rho)u_\alpha u_\beta + p g_{\alpha\beta} + u_{(\alpha}S_{\beta)\gamma}\dot{u}^\gamma - \omega^\gamma{}_{(\alpha}S_{\beta)\gamma} + u_{(\alpha}S_{\beta)}{}^\gamma \omega_{\gamma\delta}u^\delta, \tag{6}$$

where ρ is the mass density, p the pressure, and $\omega_{\alpha\beta} = e_\alpha \cdot \nabla_0 e_\beta$. Substitution of the values which the variables appearing in Eq. (6) obtain in our model leads to the form initially chosen for $t_{\alpha\beta}$, with $m = \rho(1+\epsilon)$, $p_1 = p_2 = p - 2S^2$, and $p_3 = p$.

The solution I obtained under the above assumptions reads

$$m = p = 2S^2 = 2S_0{}^2\exp(-2\mu), \qquad 2\mu = (S_0 r)^2, \quad \nu = 0, \tag{7}$$

where S_0 is an integration constant, representing the spin density at the axis $(r = 0)$ of the cylinder. All other integration constants were determined by imposing the conditions (a) that φ is a proper angular coordinate with period 2π near the axis, and (b) that the radial stress p_1 vanishes at the cylinder's surface, located at $r = R$. The latter condition will be justified shortly. Thus the interior metric becomes

$$ds^2 = -dt^2 + r^2 d\varphi^2 + \exp(-S_0{}^2 r^2)(dr^2 + dz^2). \tag{8}$$

In the vacuum region outside the cylinder the field equations reduce to those of GR, namely $R_{\alpha\beta} = R_{(\alpha\beta)} = 0$, whose general solution for the case of stationary, cylindrically symmetric metric has been obtained by Lewis.[7] It is the Lewis solution that van Stockum matched to his interior metric for the rotating dust cylinder. I use the same solution but now the matching conditions are modified. It has been shown by Arkuszewski, Copczynski, and Ponomariev[8] that the boundary conditions appropriate to the EC theory are not those of Lichnerowicz,[9] but a new set, which in our case reads, (i) the fluid particles move along the $r = R$ hypersurface, (ii) the component of stress normal to this hypersurface vanishes, (iii) the metric functions are continuous at $r = R$, and (iv)

$$\partial_r g_{\alpha\beta}|_{r=R+0} = \partial_r g_{\alpha\beta}|_{r=R-0} + 2K_{r(\alpha\beta)}, \tag{9}$$

where $\alpha, \beta \neq r$.

The first two of the above conditions have already been incorporated in our interior solution, while it is easily verified that the last two are satisfied if for $r > R$ we take

$$ds^2 = -(A^2x^{1-k} - B^2x^{1+k})dt^2 + R^2(-B^2x^{1-k} + A^2x^{1+k})d\varphi^2 - 2RAB(x^{1-k} - x^{1+k})dt\,d\varphi + \exp(-S_0^2R^2)x^{(k^2-1)/2}(dr^2 + dz^2), \tag{10}$$

where $x = r/R$, $k = (1 - 4S_0^2R^2)^{1/2}$, $A^2 = (1+k)/2k$, $B^2 = (1-k)/2k$, and $AB > 0$. I will assume that $2S_0R < 1$, since the case of imaginary k is not of interest here.

The space-time represented by the line elements (8) and (10) for the interior and exterior regions, respectively, is an exactly solvable example of the class of globally stationary but locally static space-times which was studied recently by Stachel.[10] It was shown above that the gravitational field is static in the interior region. The field is static outside the cylinder as well, since the timelike Killing vector $\xi = A\,\partial_t - BR^{-1} \times \partial_\varphi$ is hypersurface orthogonal there. However, no timelike Killing vector can be found which is hypersurface orthogonal everywhere. Equivalently, it is impossible to define a cosmic time in this space-time.

The solution presented above is distinguished by yet another feature. To my knowledge, it is the first exact solution of the EC equations in which the spin density of the source manifests itself, not simply by modifying the stress-energy tensor of the mass distribution, but by a global effect which agrees with the physical intuition about intrinsic angular momentum, namely by inducing magneticlike terms in the gravitational field. This is not the case with Prasanna's[11] class of solutions for a static cylinder in EC theory, for example.

Lastly, but most importantly, this solution provides a concrete theoretical model on the basis of which one can think of testing the EC theory vis-à-vis classical GR. Following Stachel,[10] one can consider an Aharonov-Bohm[12] type experiment in which a coherent beam of light is split into two components which are reunited after passing around opposite sides of a cylinder. A shift of the interference pattern which is produced by a rotating as well as by a static but spin-polarized cylinder would be clearly in favor of the EC theory, as only the latter can account for both cases on an equal footing.

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